Multiple Sign Changing Radially Symmetric Solutions in a General Class of Quasilinear Elliptic Equations *

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Abstract

In this paper we prove that the equation $-(r^\alpha\phi(|u'(r)|)u'(r))' = \lambda r^\gamma f(u(r)),\ 0 < r < R$, where $\alpha, \gamma, \mathbf{R}$ are given real numbers, $\phi : (0,\infty) \to (0,\infty)$ is a suitable twice differentiable function, $\lambda > 0$ is a real parameter and $f : \mathbf{R} \to \mathbf{R}$ is continuous, admits an infinite sequence of sign-changing solutions satisfying $u'(0) = u(R) = 0$. The function f is required to satisfy $tf(t) > 0$ for $t \neq 0$. Our technique explores fixed point arguments applied to suitable integral equations and shooting arguments. Our main result extends earlier ones in the case ϕ is in the form $\phi(t) = |t|^\beta$ for an appropriate constant γ.

1 Introduction

We study the nonlinear eigenvalue problem

$$\begin{cases} -(r^\alpha\phi(|u'(r)|)u'(r))' &= \lambda r^\gamma f(u(r)), \quad 0 < r < R, \\ u'(0) &= u(R) = 0, \end{cases} \tag{P_λ}$$

where $\lambda > 0$ is a parameter, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\alpha, \gamma \in \mathbb{R}$ are suitable constants.

We shall assume that $\phi : (0,\infty) \to (0,\infty)$ is a twice differentiable, C^1-function, satisfying

(ϕ_1) (i) $t\phi(t) \to 0$ as $t \to 0$,

(ii) $t\phi(t) \to \infty$ as $t \to \infty$,

(ϕ_2) $t\phi(t)$ is strictly increasing in $(0,\infty)$,

(ϕ_3) there are constants $\gamma_1, \gamma_2 > 1$ such that

$$\gamma_1 - 1 \leq \frac{(t\phi(t))'}{\phi(t)} \leq \gamma_2 - 1,\ t > 0.$$

Concerning f, the following conditions will be imposed:

(f_1) $tf(t) > 0,\ t \neq 0,$

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(f_2) there exists $d_\infty > 0$ such that f is nondecreasing in $(-\infty, d_\infty]$,

(f_3) $$\liminf_{\nu\to 0^\pm} \int_0^\nu |f(t)|^{\frac{-1}{\gamma_1-1}} \operatorname{sgn}(f(t))dt < \infty.$$

Remark 1.1. *We observe that condition (f_3) is equivalent to the following:*

(f_3') $$\max\left\{ \int_{-x}^0 [-f(t)]^{\frac{-1}{\gamma_1-1}} dt,\ \int_0^y [f(t)]^{\frac{-1}{\gamma_1-1}} dt \right\} < \infty,$$

for each $x, y > 0$, where $\gamma_1' = \gamma_1/(\gamma_1 - 1)$.

Our main objective in this work is to prove the following result:

Theorem 1.1. *Let $f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$. Assume (ϕ_1)-(ϕ_3), (f_1)-(f_2) and*

$$\gamma \geq \max\left\{\alpha, \frac{-\alpha}{\gamma_1 - 1}\right\}. \tag{γ,α}$$

Then there is a positive number Λ such that for each $\lambda \in (0, \Lambda]$, problem (P_λ) admits a positive solution u_0 and an infinite sequence $\{u_\ell\}_{\ell=1}^\infty$ of solutions satisfying:

$$u_l(0) = d_\ell, \tag{1.1}$$

$$u_\ell \text{ has precisely } \ell \text{ zeroes in } (0, R), \tag{1.2}$$

where $\{d_\ell\}_{\ell=1}^\infty$ is an infinite sequence of real numbers such that

$$d_\infty > d_1 > \cdots > d_\ell > \cdots > 0. \tag{1.3}$$

The proof of Theorem 1.1 is strongly based on the shooting method. In this regard, consider the initial value problem

$$\begin{cases} -(r^\alpha \phi(|u'(r)|)u'(r))' = \lambda r^\gamma f(u(r)), & r > 0, \\ u(0) = d, \quad u'(0) = 0, \end{cases} \tag{$P_{\lambda,d}$}$$

where $d \in (0, d_\infty]$.

The auxiliary result below will play a crucial role in this work.

Theorem 1.2. *Let $f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$. Assume (ϕ_1)-(ϕ_3), (γ, α) and (f_1)-(f_2). Then there exists a positive number $\Lambda = \Lambda(d_\infty)$ such that for each $\lambda \in (0, \Lambda]$, problem $(P_{\lambda,d})$ has a unique solituon $u(\cdot, d, \lambda) = u(\cdot, d) \in C^1([0, \infty))$. In addition, for each $d \in (0, d_\infty]$, there is a sequence $\{z_\ell\}_{\ell=1}^\infty$ of zeroes of $u(\cdot, d)$, $z_\ell = z_\ell(d)$, such that*

$$\begin{aligned} & z_1(d_\infty) \geq R, \quad u(r, d) > 0 \text{ if } 0 < r < z_1(d), \\ & z_1(d) < z_2(d) < \cdots < z_\ell(d) < \cdots, \\ & u'(r, d) < 0 \text{ if } 0 < r \leq z_1(d), u(r, d) \neq 0 \text{ if } z_\ell < r < z_{\ell+1} \text{ and } u'(z_\ell, d) \neq 0, \end{aligned} \tag{1.4}$$

$$z_\ell(d) \to 0 \text{ as } d \to 0 \text{ and } z_\ell(d) \to z_\ell(\underline{d}) \text{ as } d \to \underline{d}, \quad \underline{d} \in (0, d_\infty], \tag{1.5}$$

$$\begin{aligned} & \text{if } \underline{d} \in (0, d_\infty] \text{ and } u(\cdot, \underline{d}) \text{ has } k \text{ zeroes in } (0, R) \text{ then } u(\cdot, d) \text{ has at most } k+1 \\ & \text{zeroes in } (0, R) \text{ whenever } d < \underline{d}, d \text{ close enough to } \underline{d}. \end{aligned} \tag{1.6}$$

2 Background

Consider the problem

$$\left\{\begin{array}{rcl} -\mathrm{div}(a(x)|\nabla u(x)|^{\beta}\nabla u(x)) &=& \lambda\, b(x)f(u), \quad x\in B_R, \\ \\ u(x) &=& 0, \quad x\in\partial B_R, \end{array}\right. \tag{$P1_\lambda$}$$

where $B_R \subset \mathbf{R^N}$ is the ball of radius R centered at the origin, the functions a, b are radially symmetric and $\beta > -1$. Making $a = b \equiv 1$, $\beta = p-2$ with $1 < p < \infty$ and $\lambda = 1$, $(P1_\lambda)$ becomes

$$\left\{\begin{array}{rcl} -\left(r^{N-1}|u'(r)|^{p-2}u'(r)\right)' &=& r^{N-1}f(u(r)), \quad 0<r<R, \\ \\ u'(0)=u(R)=0. && \end{array}\right. \tag{$P2$}$$

It was shown in [6] that if $f(t) = |t|^{\delta-1}t$ with $1 < \delta+1 < p < N$ then $(P2)$ has infinitely many nodal solutions.

In [3], it was shown that the more general problem

$$\left\{\begin{array}{rcl} -\left(r^{\alpha}|u'(r)|^{\beta}u'(r)\right)' &=& \lambda r^{\gamma}f(u(r)), \quad 0<r<R, \\ \\ u'(0)=u(R)=0 && \end{array}\right. \tag{$P3$}$$

admits infinitely many solutions if λ is positive and small enough,

$$\beta > -1, \quad \gamma \geq \max\left\{\alpha, \frac{-\alpha}{\beta+1}\right\}, \tag{2.1}$$

and conditions $(f_1), (f_2)$ and a stronger form of (f_3) hold.

Regarding $(P3)$, an example of a function safistying $(f_1), (f_2), (f_3)$ with $\beta > 0$ is $f(t) = arctg(t)$.

As was pointed out by Clement, Figueiredo & Mitidieri [8] the operator

$$\left(r^{\alpha}|u'(r)|^{\beta}u'(r)\right)'$$

represents the radial form of the well known operators:

p-Laplacian with $1 < p < N$ *when* $\alpha = N-1,\ \beta = p-2,$

k-Hessian with $1 \leq k \leq N$ *when* $\alpha = N-k,\ \beta = k-1.$

The problem

$$\left\{\begin{array}{rcl} -\Delta_{\Phi}u &=& \lambda f(u) \ \text{ in } B \\ \\ u &=& 0 \ \text{ on } \partial B, \end{array}\right. \tag{Φ}$$

where

$$\Phi(t) = \int_0^t s\phi(s)ds,$$

Δ_Φ is the Φ-Laplacian operator namely

$$\Delta_\Phi u = \mathrm{div}\big(\phi(|\ \nabla u\ |)\nabla u\big),$$

and $B \subset R^N$ is the ball of radius R centered at the origin, was addressed by many authors (see e.g. Fukagai & Narukawa[4] and its references). A weak solution of (Φ) is by definition an element $u \in W_0^{1,\Phi}(B)$ (the usual Orlicz-Sobolev space) such that

$$\int_B \phi(|\nabla u|)\nabla u \nabla v dx = \lambda \int_B f(u)v dx, \; v \in W_0^{1,\Phi}(B). \tag{2.2}$$

The radially symmetric form of (Φ) is

$$\begin{cases} -(r^{N-1}\phi(|u'(r)|)u'(r))' &= \lambda r^{N-1} f(u(r)), \; 0 < r < R \\ \\ u'(0) = u(R) = 0 \end{cases}$$

which is a special case of (P_λ), (see further remarks in the Appendix).

Theorem 1.1 extends the main results of [3], [6], in the sense that we were able to treat both with a more general class of quasilinear operators and a broader class of terms f.

Problems like (P_λ) have been investigated by many authors and we would like to refer the reader to Saxton & Wei [16], Castro & Kurepa [7], Cheng [5], Strauss [14], Ni & Serrin [12], Castro, Cóssio & Neuberger [9], Fukagai & Narukawa [4], Mihailescu & Radulescu [10, 11] and their references. Here, we would point out that in [4], Fukagai & Narukawa have mentioned that this type of problem appears in some fields of physics, such as, nonlinear elasticity, plasticity and generalized Newtonian fluids.

3 Proof of Theorem 1.1

Take $\lambda \in (0, \Lambda]$ where $\Lambda > 0$ is given in Theorem 1.2. We proceed in two steps:

Step 1. **(Existence of a positive solution of (P_λ).)** Let $d \in (0, d_\infty]$. We shall use the notations in Theorem 1.2. So $z_1 = z_1(d)$ denotes the first zero of $u(\cdot) = u(\cdot, d)$. Set

$$A_0 = \Big\{ d \in (0, d_\infty] \mid z_1(d) \geq R \Big\} \text{ and } d_0 := \inf A_0.$$

By (1.4) in theorem 1.2, $z_1(d_\infty) \geq R$. So $A_0 \neq \emptyset$. We will show that

$$d_0 > 0 \text{ and } z_1(d_0) = R. \tag{3.1}$$

Indeed, assume on the contrary that $d_0 = 0$. Take a sequence $(d_j) \subseteq A_0$ such that $d_j \to 0$. By (1.5), $z_1(d_j) \to 0$, which is a contradiction.

Now, assume $z_1(d_0) > R$. Pick a sequence $(d_j) \subseteq (0, d_\infty]$ such that $d_j < d_0$ and $d_j \to d_0$. Applying (1.5) we infer that $z_1(d_j) \to z_1(d_0)$. Once $z_1(d_0) > R$, it follows that $z_1(d_j) > R$, which shows that $d_j \in A_0$. But this contradicts the definition of d_0. Therefore $z_1(d_0) = R$ and this completes the proof of (3.1). As a byproduct there is a positive solution of (P_λ).

Step 2. **(Existence of an infinite sequence of sign-changing solutions of (P_λ).)** At first consider

$$A_1 := \Big\{ d \in (0, d_0] \mid \; z_1(d) < R, \; z_2(d) \geq R \Big\} \; \text{ and } \; d_1 := \inf A_1.$$

We claim that

$$A_1 \neq \emptyset, \quad 0 < d_1 < d_0,$$
$$z_1(d_1) < R, \quad z_2(d_1) = R. \tag{3.2}$$

Let us show at first that $A_1 \neq \emptyset$. Indeed, by **Step 1** $z_1(d_0) = R$. By (1.6), if $d \in (0, d_0)$ with d close to d_0 then $u(\cdot, d)$ has at most one zero in $(0, R)$. Assume by contradiction that $u(\cdot, d)$ has no zero in $(0, R)$. Then $z_1(d) \geq R$ with $d < d_0$, impossible. It follows that $u(\cdot, d)$ has precisely one zero in $(0, R)$ and so $d \in A_1$, showing that $A_1 \neq \emptyset$.

To show that $d_1 > 0$, assume by contradiction that there is a sequence $\{d_j\} \subset A_1$ such that $d_j \to 0$. By (1.5), $z_2(d_j) \to 0$ contradicting $z_2(d_j) \geq R$.

It follows from $z_1(d_0) = R$ and definition of A_1 that $d_1 < d_0$. Therefore $0 < d_1 < d_0 \leq d_\infty$.

It remains to show that $z_1(d_1) < R$ and $z_2(d_1) = R$. To do it, let $\{d_j\} \subseteq A_1$ such that $d_j \to d_1$, so that $z_1(d_j) \to z_1(d_1) \leq R$ and $z_2(d_j) \to z_2(d_1) \geq R$.

If $z_1(d_1) = R$ then $u(\cdot, d_1)$ has no zeros in $(0, R)$. By (1.6), if $d < d_1$ and d is close to d_1, $u(\cdot, d)$ has at most one zero in $(0, R)$. If $u(\cdot, d)$ has one zero then we have $d < d_1$ and $d \in A_1$, a contradiction. On the other hand, if $u(\cdot, d)$ has no zero then $d \geq d_0 > d_1$ which is again a contradiction. Therefore, $z_1(d_1) < R$. Now, assume by contradiction that $z_2(d_1) > R$. Let $d_j \to d_1$ with $d_j < d_1$. Then, $z_1(d_j) \to z_1(d_1) < R$ and in addition, $z_2(d_j) \to z_2(d_1) > R$, so that, $z_1(d_j) < R$ and $z_2(d_j) > R$ for large j, which is impossible. Thus $z_2(d_1) = R$.

By induction, iterating the arguments above, we construct a sequence $\{d_\ell\}_{\ell=1}^{\infty} \subseteq (0, d_\infty]$ such that

$$0 < \cdots < d_\ell < \cdots < d_1 < d_\infty,$$
$$z_\ell(d_\ell) < R, \; z_{\ell+1}(d_\ell) = R, \tag{3.3}$$

with $d_\ell := \inf A_\ell$, where

$$A_\ell := \Big\{ d \in (0, d_\ell] \mid \; z_\ell(d) < R, \; z_{\ell+1}(d) \geq R \Big\}.$$

This ends the proof of step 2.

To finish the proof of Theorem 1.1, we use steps 1 and 2 to conclude that for $\lambda \in (0, \Lambda]$ the functions given by Theorem 1.2, namely $u_\ell(\cdot) = u(\cdot, d_\ell) \in C^1([0, R])$ for $\ell \geq 1$, satisfy

$$r^\alpha \phi(|u_\ell'(r)|) u_\ell'(r) \text{ is differentiable,}$$

$$-(r^\alpha \phi(|u_\ell'(r)|) u_\ell'(r))' = \lambda r^\gamma f(u_\ell(r)), \; 0 < r < R,$$

$$u_\ell'(0) = 0 \text{ and } u_\ell(R) = 0,$$

that is, u_l is a classical solution of (P_λ), u_ℓ has precisely ℓ zeroes in $(0, R)$ and so

$$u_0, u_1, u_2, \cdots,$$

is an infinite sequence of solutions of (P_λ) as claimed in the statement of Theorem 1.1. □

4 Proof of Theorem 1.2

At first we set

$$\Phi(t) = \int_0^t s\phi(s)ds, \quad H(t) = t\Phi'(t) - \Phi(t), \quad F(t) = \int_0^t f(s)ds.$$

The results below will play a crucial role in this paper.

Lemma 4.1. *Assume (γ, α) and let $d \in [0, d_\infty]$, $\lambda > 0$ and $T > 0$. If u is a solution of $(P_{\lambda,d})$ in $[0, T]$, then*

$$H(|u'(r)|) \leq \lambda r^{\gamma-\alpha}[F(d) - F(u(r))], \ r \geq 0. \tag{4.1}$$

$$F(u(r)) \leq F(d) \text{ for } r \in [0, T] \tag{4.2}$$

Lemma 4.2. *Assume that (ϕ_1)-(ϕ_3), (γ, α) and (f_1)-(f_2) holds. If $f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ and $d \in (0, d_\infty]$, then problem $(P_{\lambda,d})$ has a unique solution $u(\cdot, d, \lambda) = u(\cdot, d) = u(\cdot) \in C^1([0, \infty))$. In addition,*

$$\text{if } d_0 \in (0, d_\infty] \text{ then } u(r, d) \to u(r, d_0) \text{ as } d \to d_0, \text{ uniformly in } [0, T] \text{ for } T > 0, \tag{4.3}$$

$$\text{if } d_0 \in (0, d_\infty] \text{ then } u'(r, d) \to u'(r, d_0) \text{ as } d \to d_0, \text{ uniformly on compact subsets of } (0, \infty], \tag{4.4}$$

4.1 Proofs of Lemmas 4.1 and 4.2

Integrating the equation in $(P_{\lambda,d})$ we get to

$$\phi(|u'(r)|)u'(r) = -r^{-\alpha} \int_0^r \lambda t^\gamma f(u(t))dt, \ r > 0. \tag{4.5}$$

Setting

$$h(t) := t\phi(t), \tag{4.6}$$

we see that h is invertible with differentiable inverse h^{-1}. Then,

$$u'(r) = h^{-1}\left(-r^{-\alpha} \int_0^r \lambda t^\gamma f(u(t))dt\right) \text{ if } \ u'(r) > 0, \tag{4.7}$$

$$u'(r) = -h^{-1}\left(-r^{-\alpha} \int_0^r \lambda t^\gamma f(u(t))dt\right) \quad \text{if } \ u'(r) < 0, \tag{4.8}$$

Once f is continuous and $\gamma \geq \alpha$, we conclude from the above equalities that $u \in C^1$.

Proof of Lemma 4.1. From (4.7) and (4.8), we infer that $u \in C^2$ at the points $r > 0$ where $u'(r) \neq 0$. Computing derivatives in $(P_{\lambda,d})$ and multiplying the resulting equality by $u'(r)$, we are led to

$$-\alpha r^{\alpha-1}\phi(|u'(r)|)|u'(r)|^2 - r^\alpha \frac{d}{dt}h(|u'(r)|)u'(r)u''(r) = \lambda r^\gamma f(u(r))u'(r), \ u'(r) \neq 0. \tag{4.9}$$

Consider the functional $E : [0, \infty) \to \mathbb{R}$ defined by

$$E(0) = \lambda F(d) \text{ and } E(r) = r^{\alpha-\gamma}[H(|u'(r)|)] + \lambda F(u(r)),\ r > 0,$$

where $H(t) = t\Phi'(t) - \Phi(t) = \int_0^t h'(s)s\ ds$. Note that

$$E'(r) = r^{\alpha-\gamma}[H(|u'(r)|)]' + (\alpha - \gamma)r^{\alpha-\gamma-1}H(|u'(r)|) + \lambda f(u(r))u'(r),\ r > 0$$

and

$$[H(|u'(t)|)]' = \frac{d}{dt}h(|u'(r)|)u'(r)u''(r),\ u'(r) \neq 0.$$

Therefore from (4.9),

$$E'(r) = (\alpha - \gamma)r^{\alpha-\gamma-1}H(|u'(r)|) - \alpha r^{\alpha-\gamma-1}\phi(|u'(r)|)|u'(r)|^2\ u'(r) \neq 0.$$

From Lemma 5.6 in the Appendix, the last inequality combined with hypothesis (γ, α) gives

$$E'(r) \leq \frac{\gamma_1 - 1}{\gamma_1}(\alpha - \gamma)r^{\alpha-\gamma-1}\phi(|u'(r)|)|u'(r)|^2 - \alpha r^{\alpha-\gamma-1}\phi(|u'(r)|)|u'(r)|^2 < 0,\ u'(r) \neq 0. \quad (4.10)$$

Next, we will prove that E is continuous at the origin and therefore, as E is non-decreasing by the previous inequality, it follows that $E(r) \leq E(0)$ for all $r \geq 0$. Note that equation (4.5) implies

$$\Phi(|u'(r)|) = \Phi\left(h^{-1}\left(\left|r^{-\alpha}\int_0^r \lambda t^\gamma f(u(t))dt\right|\right)\right),$$

which in turn gives

$$\Phi(|u'(r)|) \leq \Phi\left(h^{-1}\left(\frac{\lambda C_{\delta,d}r^{\gamma-\alpha+1}}{\gamma+1}\right)\right),\ r \in [0, \delta), \quad (4.11)$$

where $C_{\delta,d} = \max_{r\in[0,\delta]}|f(u(r))|$. We choose $\delta > 0$ small and apply Lemmas 5.1 and 5.2 to conclude from (4.11) that

$$\Phi(|u'(r)|) \leq \left(\frac{\lambda C_{\delta,d}}{\gamma+1}\right)^{\frac{\gamma_2}{\gamma_1-1}} r^{\frac{\gamma_2}{\gamma_1-1}(\gamma-\alpha+1)}, \forall\ r \in [0, \delta). \quad (4.12)$$

We apply condition Δ_2 (see inequality (5.1) in the Appendix) in the definition of E to infer that

$$E(r) \leq (\gamma_2 - 1)r^{\alpha-\gamma}\Phi(|u'(r)|) + \lambda F(u(r)),\ r > 0. \quad (4.13)$$

Thus, (4.12) and (4.13) lead to

$$E(r) \leq Cr^{(\alpha-\gamma)+\frac{\gamma_2}{\gamma_1-1}(\gamma-\alpha+1)} + \lambda F(u(r)),\ r \in [0, \delta). \quad (4.14)$$

Recalling that $\gamma_2 \geq \gamma_1$, we have that

$$(\alpha - \gamma) + \frac{\gamma_2,}{\gamma_1 - 1}(\gamma - \alpha + 1) \geq \frac{\gamma_1 - \alpha + \gamma}{\gamma_1 - 1} > 0.$$

Hence, from (4.14) that $\lim\limits_{r\to 0} E(r) \leq \lambda F(d)$. On the other hand, by Lemma 5.6, we know that $H(t) \geq 0$ for all $t \geq 0$. Then, by definition of E, $E(r) \geq \lambda F(u(r))$ for all $r > 0$. Gathering these information, we conclude that

$$\lim_{r\to 0} E(r) = \lambda F(d).$$

Therefore, as (4.10) is true,

$$E(r) \le E(0) \text{ for } r \ge 0,$$

which is equivalent to the desired inequality namely (4.1) $\square$

Proof of Lemma 4.2 We will at first study existence and uniqueness of local solutions of $(P_{\lambda,d})$. Let $\epsilon > 0$ and consider

$$\begin{cases} -(r^\alpha \phi(|u'(r)|)u'(r))' = \lambda r^\gamma f(u(r)), & 0 < r < \epsilon, \\ \\ u(0) = d, \quad u'(0) = 0. \end{cases} \tag{$P_{\lambda,d,\epsilon}$}$$

We shall need the following result whose proof is left to the Appendix.

Lemma 4.3. *$(P_{\lambda,d,\epsilon})$ has a unique solution $u(\cdot) = u(\cdot, d, \lambda, \epsilon) \in C^2([0,\epsilon))$ for small ϵ.*

Proof of Uniqueness in Lemma 4.2 Assume that u, v are two $C^1([0,\infty))$ solutions. Let

$$S_0 = \{r \ge 0 : \; u(t) = v(t), \; 0 \le t \le r\}.$$

We will show that

$$S_0 \ne \emptyset, \; S_0 \text{ is both open and closed in } [0,\infty). \tag{4.15}$$

Indeed, by Lemma 4.3 above, $[0,\epsilon) \subset S_0$ for $\epsilon > 0$ small enough, which shows that $S_0 \ne \emptyset$. Moreover, since u, v are C^1 functions we infer that S_0 is closed. To finish we shall prove that S_0 is open. To achieve that let $\widehat{r} \in S_0$ with $\widehat{r} > 0$. We distinguish between two cases.

Case 1. $u'(\widehat{r}) = v'(\widehat{r}) = 0$

Assume $u(\widehat{r}) = v(\widehat{r}) = \widehat{d}$. If $\widehat{d} = 0$ then, up to a translation in the domain, we are within the settings of Lemma 4.1. Therefore, using (4.1), observing that by hypothesis (f_1) one has $F(u(r)) \ge 0$ for $r \ge \widehat{r}$, and noticing that $F(\widehat{d}) = 0$, we get

$$H(|u'(r)|) \le \lambda r^{\gamma-\alpha}\left(F(\widehat{d}) - F(u(r))\right) \le 0 \text{ for } r \ge \widehat{r},$$

from where it follows that $u(r) = 0$ for $r \ge \widehat{r}$, because by Lemma 5.6 in the Appendix

$$H(t) \ge 0 \;\; \forall t \ge 0 \;\text{ and }\; H(t) = 0 \Leftrightarrow t = 0.$$

The same argument works to prove that $v(r) = 0$ for $r \ge \widehat{r}$. Consequently, $r \ge \widehat{r}$, $u(r) = v(r) = 0$ and then, $S_0 = [0,\infty)$ is open. On the other hand, if $\widehat{d} > 0$, we define

$$\widehat{K}^\epsilon_\rho(\hat{d}) = \{u \in C([\widehat{r}, \widehat{r}+\epsilon]) : \; u(0) = \widehat{d}, \; \|u - \widehat{d}\|_\infty \le \rho\},$$

$$\widehat{T}(u(r)) = \widehat{d} - \int_{\widehat{r}}^r h^{-1}\left(t^{-\alpha}\int_{\widehat{r}}^t \lambda\tau^\gamma f(u(\tau))d\tau\right)dt, \; \forall\, r \in [0,\epsilon],$$

where ϵ, ρ are positive and ϵ is small. The same proofs of (5.8) and (5.9) can be do here, and then the Banach Fixed Point Theorem guarantees a unique fixed point for the operator $\widehat{T}$ when ϵ is small, therefore, $u(r) = v(r)$ in a small neighbourhood of $\widehat{r}$, which implies that S_0 is open.

Case 2. $u'(\widehat{r}) = v'(\widehat{r}) \ne 0$.

Note that there is a neighbourhood V of $\widehat{r}$ such that $u'(r), v'(r) \neq 0$ for $r \in V$. So in V, we must conclude, as in (4.10) (here we use the same notation as in the proof of Lemma 4.1) that if z denotes either u or v then

$$\left(r^{\alpha-\gamma} H(|z'(r)|) + \lambda F(z(r))\right)' = -\frac{\alpha+\gamma(\gamma_1-1)}{\gamma_1} r^{\alpha-\gamma-1}\phi(|z'(r)|)z'(r)|^2.$$

Integrating from $\widehat{r}$ to r and subtracting the corresponding equations for $z = u$ and $z = v$, we obtain (remember that $u(\widehat{r}) = v(\widehat{r})$ and $u'(\widehat{r}) = v'(\widehat{r})$)

$$\begin{aligned} &r^{\alpha-\gamma}[H(|u'(r)|) - H(|v'(r)|)] + \lambda\,[F(u(r)) - F(v(r))] = \\ &-\frac{\alpha+\gamma(\gamma_1-1)}{\gamma_1}\int_{\widehat{r}}^{r} t^{\alpha-\gamma-1}\left[\phi(|u'(t)|)|u'(t)|^2 - \phi(|v'(t)|)|v'(t)|^2\right] dt. \end{aligned} \tag{4.16}$$

Next we consider three auxliary continuous functions, namely

$$A_1(t) = \begin{cases} \frac{H(|u'(t)|)-H(|v'(t)|)}{u'(t)-v'(t)}, & u'(t) \neq v'(t) \\ \\ \phi(|u'(t)|)u'(t), & u'(t) = v'(t), \end{cases}$$

$$A_2(t) = \begin{cases} \dfrac{h(|u'(t)|)|u'(t)| - h(|v'(t)|)|v'(t)|}{u'(t)-v'(t)}, & u'(t) \neq v'(t) \\ \\ \frac{d}{dt}[h(|u'(t)|)|u'(t)|], & u'(t) = v'(t), \end{cases}$$

$$B(t) = \begin{cases} \lambda\frac{F(u(t))-F(v(t))}{u(t)-v(t)}, & u(t) \neq v(t) \\ \\ \lambda f(u(t)), & u(t) = v(t). \end{cases}$$

Let $w(r) = u(r) - v(r)$. From (4.16),

$$r^{\alpha-\gamma} A_1(r) w'(r) + B(r) w(r) = -\frac{\alpha+\gamma(\gamma_1-1)}{\gamma_1}\int_{\hat{r}}^{r} t^{\alpha-\gamma-1} A_2(t) w'(t) dt. \tag{4.17}$$

Once $u'(\widehat{r}) \neq 0$, we have that in a neighbourhood of $\widehat{r}$, the function $1/A_1$ is well defined and continuous, and so, equation (4.17) is the same as

$$w'(r) + \frac{B(r)}{A_1(r)} r^{\gamma-\alpha} w(r) = -\frac{\alpha+\gamma(\gamma_1-1)}{\gamma_1}\frac{r^{\gamma-\alpha}}{A_1(r)}\int_{\hat{r}}^{r} t^{\alpha-\gamma-1} A_2(t) w'(t) dt. \tag{4.18}$$

As h is two times differentiable and $u'(\widehat{r}) \neq 0$, A_2 is continuously differentiable in a neighborhood of $\widehat{r}$, therefore, from (4.18) and integration by parts we obtain

$$\begin{aligned} w'(r) + \frac{B(r)}{A_1(r)} r^{\gamma-\alpha} w(r) = \;& \frac{\alpha+\gamma(\gamma_1-1)}{\gamma_1}\frac{r^{\gamma-\alpha}}{A_1(r)} r^{\alpha-\gamma-1} A_2(r) w(r) + \\ & -\frac{\alpha+\gamma(\gamma_1-1)}{\gamma_1}\frac{r^{\gamma-\alpha}}{A_1(r)}\int_{\widehat{r}}^{r}\left[t^{\alpha-\gamma-1}A_2(t)\right]' w(t) dt, \end{aligned}$$

hence

$$
\begin{aligned}
w'(r)+&\left[\frac{B(r)}{A_1(r)}r^{\gamma-\alpha}-\frac{\alpha+\gamma(\gamma_1-1)}{\gamma_1}\frac{r^{\gamma-\alpha}}{A_1(r)}r^{\alpha-\gamma-1}A_2(r)\right]w(r)= \\
&-\frac{\alpha+\gamma(\gamma_1-1)}{\gamma_1}\frac{r^{\gamma-\alpha}}{A_1(r)}\int_{\widehat{r}}^{r}\left[t^{\alpha-\gamma-1}A_2(t)\right]'w(t)dt.
\end{aligned}
\tag{4.19}
$$

We introduce the notation

$$
D_1(r)=\frac{B(r)}{A_1(r)}r^{\gamma-\alpha}-\frac{\alpha+\gamma(\gamma_1-1)}{\gamma_1}\frac{r^{\gamma-\alpha}}{A_1(r)}r^{\alpha-\gamma-1}A_2(r),
$$

$$
D_2(r)=-\frac{\alpha+\gamma(\gamma_1-1)}{\gamma_1}\frac{r^{\gamma-\alpha}}{A_1(r)},
$$

$$
D_3(r)=\left[t^{\alpha-\gamma-1}A_2(t)\right]',
$$

which implies from (4.19) that

$$
w'(r)+D_1(r)w(r)=D_2(r)\int_{\hat{r}}^{r}D_3(s)w(s)ds. \tag{4.20}
$$

We integrate equation (4.20) from $\widehat{r}$ to r, which combined with the fact that, $A_1, A_2, 1/A_1, A_2', B$ are bounded functions (remember they are all continuous functions in a neighborhood of $\hat{r}$) to conclude that

$$
\begin{aligned}
|v(r)| &\leq \int_{\widehat{r}}^{r}|D_1(s)||v(s)|ds+\int_{\widehat{r}}^{r}|D_2(s)|\int_{\widehat{r}}^{s}|D_3(r)||v(t)|dtds \\
&\leq C\int_{\widehat{r}}^{r}|v(s)|ds,
\end{aligned}
$$

where $C>0$ is a constant. By the Gronwall Inequality, $v=0$ in a neighborhood of $\hat{r}$. Therefore, S_0 is open and (4.15) is proved.

Proof of Existence in Lemma 4.2. Let

$$
S_\infty=\{r>0 \mid (P_{\lambda,d}) \text{ has a solution in } [0,r)\} \text{ and } T_\infty=\sup S_\infty.
$$

We will prove that

$$
T_\infty=\infty. \tag{4.21}
$$

Assume, on the contrary, that $T_\infty<\infty$. First note that S_∞ is a closed set. Indeed, let $r_n\to r$ with $r_n\in S_\infty$. If $r<r_n$ for some n then $r\in S_\infty$ by force, so we can assume that $r_n<r$ and without loss of generality that $r_n<r_{n+1}$. If u_n are the solutions associated with r_n, we define $u:[0,r)\to\mathbb{R}$ by $u(x):=u_n(x)$ of $x\in[0,r_n)$. Once (4.15) is satisfied, we conclude that u is well defined and it is a solution of $(P_{\lambda,d})$, which implies that $r\in S_\infty$.

Since S_∞ is closed, we have that $T_\infty\in S_\infty$. Let u be the solution associated with T_∞. We first observe that from (4.1), $|u'|$ is bounded, which implies that u can be continuously extended to T_∞. Moreover, equation (4.5) guarantees that $u'(T_\infty)$ is uniquely defined, so there are two cases to consider.:

Case 1. $u'(T_\infty)=0$.

If $u(T_\infty)=0$, consider the extension of u namely $\widetilde{u}:[0,\infty)\to\mathbb{R}$ given by $\widetilde{u}(t)=0$ for $t\geq T_\infty$. Then $\widetilde{u}$ is a C^1 function and it is also a solution of $(P_{\lambda,d})$, which is an absurd. Otherwise, if $u(T_\infty)=d^\infty>0$, consider the operator T defined by

$$T(u(r))=d^\infty-\int_{T_\infty}^{r}h^{-1}\left(t^{-\alpha}\int_{T_\infty}^{t}\lambda\tau^\gamma f(u(\tau))d\tau\right)dt.$$

Following the same lines as in the proof of either (5.8) or (4.15) **Case 1**, we have that T has unique fixed point $v:[0,T_\infty+\epsilon]$, which is an absurd due to the definition of T_∞.

Case 2. $u'(T_\infty)\neq 0$.
Assume without loss of generality that $u'(T_\infty)>0$. Then, by (4.5),

$$u'(r)=h^{-1}\left(r^{-\alpha}\int_0^r\lambda t^\gamma f(u(t))dt\right)$$

in a neighborhood of T_∞. Hence, u is C^2 in a neighborhood of T_∞, which implies by $(P_{\lambda,d})$ that

$$u''(r)=-\left[\frac{d}{dt}h(u'(r))\right]^{-1}\left(\frac{\alpha}{r}\phi(u'(r))u'(r)+\lambda r^{\gamma-\alpha}f(u(r))\right).$$

By the last equation and Peano's Theorem, u can be extended to $[0,T_\infty+\delta)$, where $\delta>0$ and thus we reach an absurd, because such extension is also a solution to $(P_{\lambda,d})$. This finishes the proof of **Case 2**. Therefore, (4.21) is proved and thus Claim 2 is also proved.
Proof of (4.3). Remember that

$$r^\alpha\phi(|u'(r)|)u'(r)=-\int_0^r\lambda t\gamma f(u(t))dt. \tag{4.22}$$

Assume that $d_n\to d_0$ and set $u_n(r)=u(r,d_n)$, $u_0(r)=u(r,d_0)$. Inequality (4.5) implies that $|u_n'(r)|$ is bounded for $r\in[0,T]$, therefore, by Áscoli-Arzéla Theorem, there is a subesequence, still denoted by u_n, such that $u_n\to v$ uniformly in $[0,T]$ for some $v\in C([0,T])$. Now we will prove that $v=u_0$.
First note that by Lebesgue's Theorem

$$\int_0^r\lambda t\gamma f(u_n(t))dt\to\int_0^r\lambda t\gamma f(v(t))dt,$$

and by (4.22),

$$r^\alpha\phi(|u_n'(r)|)u_n'(r)\to-\int_0^r\lambda t\gamma f(v(t))dt,\ r\in[0,T].$$

As a consequence,

$$|u_n'(r)|\to h^{-1}\left(r^{-\alpha}\left|\int_0^r\lambda t\gamma f(v(t))dt\right|\right),\ r\in[0,T]. \tag{4.23}$$

The combination of (4.22) and (4.23) implies that $u_n'(r)\to w(r)$ for all $r\in[0,T]$ where w is a continuous function. Hence, applying Lebesgue's Theorem we obtain

$$u_n(r)-d_n=\int_0^r u_n'(t)dt\to\int_0^r w(t)dr,\ r\in[0,T],$$

which implies that $w(r) = v'(r)$ and $v'(0) = 0$. Once

$$\phi(|v'(r)|)v'(r) = -r^{-\alpha} \int_0^r \lambda t^\gamma f(v(t))dt,$$

is satisfied and since $v(0) = d_0$, it follows by the uniqueness of solutions given by theorem 4.2 that $v = u_0$, which concludes the proof of (4.3).

Proof of (4.4). Let $0 < a \le r \le b < \infty$ and assume that $d_n \to d_0$. By (4.22),

$$r^\alpha |\phi(|u_n'(r)|)u_n'(r) - \phi(|u_0'(r)|)u_0'(r)| \le \int_0^r \lambda t^\gamma |f(u_n(t)) - f(u_0(t))|dt.$$

Since (u_n) converges uniformly to u_0 in $[a, b]$, we conclude from the previous inequality that

$$(\phi(|u_n'(r)|)u_n'(r) - \phi(|u_0'(r)|)u_0'(r))(u_n'(r) - u_0'(r)) \to 0,$$

uniformly in $[a, b]$. Now, we combine a generalized form of Simon's inequality, see Lemma 5.5 in the Appendix, with the last convergence to conclude that $u_n' \to u_0'$ uniformly in $[a, b]$. This finishes the proof of Lemma 4.2. □

4.2 Proof of Theorem 1.2 (Continued)

Proof of (1.4). We will start by proving that there is $z_1 = z_1(d) > 0$ such that $u(z_1) = 0$, $u'(z_1) < 0$ and

$$u(r) > 0,\ u'(r) < 0 \text{ for } 0 < r < z_1. \tag{4.24}$$

Suppose, on the contrary, that $u(r) > 0$ for all $r > 0$. It follows from (4.22) and conditions (f_1), (f_2) that $u'(r) < 0$ and

$$-u'(r) \ge h^{-1}\left(\lambda \frac{r^{\gamma-\alpha+1}}{\gamma+1} f(u(r))\right),\ r > 0.$$

Note that $u'(r) \to 0$ if $r \to \infty$ because $u(r) > 0$. Hence, the previous inequality implies that $u(r) \to 0$ if $r \to \infty$. Moreover, by Lemma 5.1 and the previous inequality, we also obtain

$$-u'(r) \ge \max\left\{ \left(\frac{\lambda r^{\gamma-\alpha+1} f(u(r))}{(\gamma+1)h(1)}\right)^{\frac{1}{\gamma_1-1}}, \left(\frac{\lambda r^{\gamma-\alpha+1} f(u(r))}{(\gamma+1)h(1)}\right)^{\frac{1}{\gamma_2-1}} \right\}, r > 0,$$

which implies

$$-u'(r) \min\{f(u(r))^{\frac{-1}{\gamma_1-1}}, f(u(r))^{\frac{-1}{\gamma_1-2}}\} \ge \min\left\{ \left(\frac{\lambda r^{\gamma-\alpha+1}}{(\gamma+1)h(1)}\right)^{\frac{1}{\gamma_1-1}}, \left(\frac{\lambda r^{\gamma-\alpha+1}}{(\gamma+1)h(1)}\right)^{\frac{1}{\gamma_2-1}} \right\}$$

for each $r > 0$. We integrate the last inequality from 0 to r and apply the change of variables $t = u(s)$ to conclude that

$$\int_{u(r)}^d \min\{f(t)^{\frac{-1}{\gamma_1-1}}, f(t)^{\frac{-1}{\gamma_2-1}}\}dt \ge \int_0^r \min\left\{ \left(\frac{\lambda s^{\gamma-\alpha+1}}{(\gamma+1)h(1)}\right)^{\frac{1}{\gamma_1-1}}, \left(\frac{\lambda s^{\gamma-\alpha+1}}{(\gamma+1)h(1)}\right)^{\frac{1}{\gamma_2-1}} \right\} ds. \tag{4.25}$$

Hypothesis (γ, α) implies that the right hand side of (4.25) converges to infinity as $r \to \infty$. Therefore, (4.25) yields

$$\liminf_{r\to\infty} \int_{u(r)}^d \min\{f(t)^{\frac{-1}{\gamma_1-1}}, f(t)^{\frac{-1}{\gamma_2-1}}\}dt = \infty,$$

which combined with (f_1) and the fact that $u(r) \to 0$ if $r \to \infty$, implies a contradiction to (f_3) and thus, (4.24) is true. To proceed, we will prove that there is $\Lambda > 0$ such that

$$z_1(d_\infty, \lambda) \geq R \text{ if } 0 < \lambda \leq \Lambda. \tag{4.26}$$

Indeed, by (4.22),

$$-u'(r) \leq h^{-1}\left(\frac{\lambda f(d_\infty) r^{\gamma-\alpha+1}}{\gamma+1}\right) \text{ for } r \in [0, z_1(d_\infty, \lambda)].$$

Integrating from 0 to $r \in [0, d_\infty]$ and making use of Lemma 5.1, we get that

$$\begin{aligned} -u(r) + d_\infty \leq \max\Bigg\{ &(\gamma_1 - 1)\left(\frac{\lambda f(d_\infty)}{(\gamma+1)h(1)}\right)^{\frac{1}{\gamma_1-1}} \frac{r^{\frac{\gamma-\alpha+\gamma_1}{\gamma_1-1}}}{\gamma-\alpha+\gamma_1}, \\ &(\gamma_2 - 1)\left(\frac{\lambda f(d_\infty)}{(\gamma+1)h(1)}\right)^{\frac{1}{\gamma_2-1}} \frac{r^{\frac{\gamma-\alpha+\gamma_2}{\gamma_2-1}}}{\gamma-\alpha+\gamma_2}\Bigg\}. \end{aligned} \tag{4.27}$$

Let $\nu \in (0,1)$. Choose $r_\infty(\nu) \in (0, z_1(d_\infty, \lambda))$ such that $u(r_\infty(\nu), d_\infty) = \nu d_\infty$. Set $r = r_\infty(\nu)$ in (4.27) and choose the maximum value in the right hand side of (4.27) which actually is

$$(\gamma_1 - 1)\left(\frac{\lambda f(d_\infty)}{(\gamma+1)h(1)}\right)^{\frac{1}{\gamma_1-1}} \frac{r_\infty(\nu)^{\frac{\gamma-\alpha+\gamma_1}{\gamma_1-1}}}{\gamma-\alpha+\gamma_1}.$$

Take $R > 0$ and choose $\Lambda_\nu > 0$ satisfying

$$1 - \nu = \left[\left(\frac{\lambda f(d_\infty)}{(\gamma+1)h(1)}\right)^{\frac{1}{\gamma_1-1}} \frac{\gamma_1 - 1}{\gamma-\alpha+\gamma_1}\right]^{-1} \frac{\Lambda_\nu^{\frac{1}{\gamma_1-1}} R^{\frac{\gamma-\alpha+\gamma_1}{\gamma_1-1}}}{d_\infty}. \tag{4.28}$$

We infer from (4.27) and (4.28) that

$$\left[\left(\frac{\lambda f(d_\infty)}{(\gamma+1)h(1)}\right)^{\frac{1}{\gamma_1-1}} \frac{\gamma_1 - 1}{\gamma-\alpha+\gamma_1}\right]^{-1} \Lambda_\nu^{\frac{1}{\gamma_1-1}} R^{\frac{\gamma-\alpha+\gamma_1}{\gamma_1-1}} \leq (\gamma_1 - 1)\left(\frac{\lambda f(d_\infty)}{(\gamma+1)h(1)}\right)^{\frac{1}{\gamma_1-1}} \frac{r_\infty(\nu)^{\frac{\gamma-\alpha+\gamma_1}{\gamma_1-1}}}{\gamma-\alpha+\gamma_1},$$

which implies that

$$\Lambda_\nu^{\frac{1}{\gamma_1-1}} R^{\frac{\gamma-\alpha+\gamma_1}{\gamma_1-1}} \leq \lambda^{\frac{1}{\gamma_1-1}} r_\infty(\nu)^{\frac{\gamma-\alpha+\gamma_1}{\gamma_1-1}}.$$

Hence,

$$R \leq r_\infty(\nu) \leq z_1(d_\infty, \lambda) \text{ if } 0 < \lambda \leq \Lambda_\nu. \tag{4.29}$$

To finish the proof of (4.26), first note that the maximum of two continuous functions is a continuous function. Therefore, (4.27) combined with (4.28) gives

$$\Lambda_\nu^{\frac{1}{\eta-1}} \overset{\nu\to 0}{\longrightarrow} \left(\frac{\lambda f(d_\infty)}{(\gamma+1)h(1)}\right)^{\frac{1}{\eta-1}} \frac{\eta - 1}{\gamma-\alpha+\eta} \frac{d_\infty}{R^{\frac{\gamma-\alpha+\eta}{\eta-1}}},$$

where either $\eta = \gamma_1$ or $\eta = \gamma_2$ depending on whether the maximum in (4.27) is assumed at γ_1 or γ_2. Note also that $r_\nu(d_\infty)$ is continuous on ν and $r_\nu(d_\infty) \to z_1(d_\infty, \lambda)$ as $\nu \to 0$, therefore, we conclude from (4.29) that

$$R \leq z_1(d_\infty, \lambda) \text{ if } 0 < \lambda \leq \Lambda,$$

where

$$\Lambda := \frac{\lambda f(d_\infty)}{(\gamma+1)h(1)} \left(\frac{\eta-1}{\gamma-\alpha+\eta}\right)^{\eta-1} \frac{d_\infty^{\eta-1}}{R^{\gamma-\alpha+\eta}}.$$

Now we will show that there is $z_2 = z_2(d) > z_1$ such that $u(z_2) = 0$, $u'(z_2) > 0$ and

$$u(r) < 0, \; z_1 < r < z_2. \tag{4.30}$$

In fact, since $u'(z_1) < 0$ then, $u'(r) < 0$ in a neighborhood of z_1. We start by proving that there is $m_1 > z_1$ such that $u'(m_1) = 0$. Thus, suppose by contradiction that it is not true, i.e. $u'(r) < 0$ for all $r > z_1$. We have by (4.2) that

$$\int_0^{u(r)} f(t)dt \leq F(d), \; r \geq 0.$$

If there is some sequence $r_n \to \infty$ such that $u(r_n) \to -\infty$ then, by the previous inequality we infer that

$$\int_{-\infty}^0 f(s)ds = \lim_n \int_{u(r_n)}^0 f(s)ds \geq -F(d),$$

which is impossible, because (f_1), (f_2) imply that $\int_{-\infty}^0 f(s)ds = -\infty$. Hence, there is $C > 0$ such that

$$u(r) \geq -C, \; u'(r) < 0, \; \forall r \geq z_1,$$

and consequently $u(r) \to L$ as $r \to \infty$ for some $L < 0$. Now, by (4.1),

$$\frac{\Phi(|u'(r)|)}{r^{\gamma-\alpha+1}} \to 0 \text{ as } r \to \infty,$$

which implies by using the inequality $\Phi(s) \geq cs^2\phi(s)$ that

$$\frac{\phi(|u'(r)|)}{r^{\gamma-\alpha+1}} \to 0.$$

On one hand (4.22) and the previous limits imply that

$$\frac{1}{r^{\gamma+1}} \int_0^r t^\gamma f(u(t))dt \to 0,$$

and on the other side, (f_1) and L'Hospital rule imply that

$$\lim_{r\to\infty} \frac{1}{r^{\gamma+1}} \int_0^r t^\gamma f(u(t))dt = \lim_{r\to\infty} \frac{r^\gamma f(u(r))}{(\gamma+1)r^\gamma} = \frac{f(L)}{\gamma+1} < 0,$$

which is an absurd. Therefore, $u'(m_1) = 0$ for some $z_1 < m_1$, so that

$$u(r) < 0 \text{ for } z_1 < r < m_1 \text{ and } u'(r) < 0 \text{ for } z_1 \leq r < m_1.$$

Now, taking $r > m_1$, r close to m_1 we have

$$\int_{m_1}^r t^\gamma f(u(t)) < 0,$$

which implies by (4.22) that

$$u(r) < 0, \; u'(r) > 0 \text{ for all } r > m_1, \; r \text{ close to } m_1.$$

Assume by contradiction that $u(r) < 0$ for $r > m_1$, so that $u'(r) > 0$. Since by (f_2)

$$-r^{\alpha}\phi(|u'(r)|)u'(r) = \lambda \int_{m_1}^{r} t^{\gamma} f(u(t))dt \leq \frac{\lambda f(u(r))}{\gamma+1}(r^{\gamma+1} - m_1^{\gamma+1}),$$

we get by taking $r > \overline{r} = 2^{\frac{1}{\gamma+1}} m_1$ above, that $r^{\gamma+1} - m_1^{\gamma+1} > \frac{r^{\gamma+1}}{2}$ and so

$$-r^{\alpha}\phi(|u'(r)|)u'(r) \leq \frac{\lambda f(u(r))}{2(\gamma+1)} r^{\gamma+1},$$

which, combined with Lemma 5.1 gives

$$u'(r) \geq \min\left\{ \left(\frac{-\lambda f(u(r))}{2(\gamma+1)} r^{\gamma-\alpha+1} \right)^{\frac{1}{\gamma_1-1}}, \left(\frac{-\lambda f(u(r))}{2(\gamma+1)} r^{\gamma-\alpha+1} \right)^{\frac{1}{\gamma_2-1}} \right\}, \ r > \overline{r}. \tag{4.31}$$

Integrating in (4.31) from $\overline{r}$ to r, we have

$$\int_{\overline{r}}^{r} u'(t) \max\{(-f(u(t)))^{\frac{-1}{\gamma_1-1}}, (-f(u(t)))^{\frac{-1}{\gamma_2-1}}\}dt \geq \int_{\overline{r}}^{r} \min\left\{ \left(\frac{t^{\gamma-\alpha+1}}{2(\gamma+1)} \right)^{\frac{1}{\gamma_1-1}}, \left(\frac{t^{\gamma-\alpha+1}}{2(\gamma+1)} \right)^{\frac{1}{\gamma_2-1}} \right\} dt,$$

for $r > \overline{r}$. Making the change of variables $y = u(t)$,

$$\int_{u(\overline{r})}^{u(r)} \max\{(-f(t))^{\frac{-1}{\gamma_1-1}}, (-f(t))^{\frac{-1}{\gamma_2-1}}\}dt \geq \int_{\overline{r}}^{r} \min\left\{ \left(\frac{t^{\gamma-\alpha+1}}{2(\gamma+1)} \right)^{\frac{1}{\gamma_1-1}}, \left(\frac{t^{\gamma-\alpha+1}}{2(\gamma+1)} \right)^{\frac{1}{\gamma_2-1}} \right\} dt. \tag{4.32}$$

Once $u(r) < 0$ and $u'(r) > 0$ for $r > \overline{r}$ it follows that $u'(r) \to 0$ as $r \to \infty$. Hence, inequality (4.31) implies that $u(r) \to 0$ as $r \to \infty$. Moreover, the right hand side of (4.32) converges to ∞ due to hypothesis (γ, α). Therefore

$$\liminf \int_{u(\overline{r})}^{u(r)} \max\{(-f(t))^{\frac{-1}{\gamma_1-1}}, (-f(t))^{\frac{-1}{\gamma_2-1}}\} = \infty,$$

which contradicts (f_3), so (4.30) is proved. Now we will prove that there is $z_3 = z_3(d) > z_2$ such that $u(z_3) = 0$, $u'(z_3) < 0$ and

$$u(r) > 0 \text{ for all } r \in (z_2, z_3). \tag{4.33}$$

Indeed, since by (4.30), $u'(z_2) > 0$, so that

$$u'(r) > 0 \text{ for all } r > z_2, \ r \text{ close to } z_2.$$

We claim that there is $m_2 > z_2$ such that $u'(m_2) = 0$. In fact, othewise, $u'(r) > 0$, for all $r > z_2$, which gives that $u(r) > 0$ for $r > z_2$. By (4.2),

$$\int_0^{u(r)} f(t)dt \leq \int_0^d f(t)dt,$$

so that $u(r) \leq d$ for $r \geq z_2$. Hence, there is $L \in (0, d]$ such that

$$u(r) \to L \text{ and } u(r) \leq L, \ r \geq z_2.$$

As in the proof of (4.30),

$$\frac{1}{r^{\gamma+1}}\int_0^r t^\gamma f(u(t))dt \to 0,$$

and

$$\lim_{r\to\infty}\frac{1}{r^{\gamma+1}}\int_0^r t^\gamma f(u(t))dt = \lim_{r\to\infty}\frac{r^\gamma f(u(r))}{(\gamma+1)r^\gamma} = \frac{f(L)}{\gamma+1} < 0,$$

which is an absurd. As a consequence, there is $m_2 > z_2$ such that $u'(m_2) = 0$ and $u'(r) > 0$, $z_2 \le r < m_2$, proving the claim. Assume again, by contradiction, that $u(r) > 0$ for all $r > m_2$ so that $u'(r) < 0$ also for all $r > m_2$. We have (similar to the proof of (4.30))

$$-r^\alpha \phi(|u'(r)|)u'(r) = \lambda \int_{m_2}^r t^\gamma f(u(t))dt \ge \frac{\lambda f(u(r))}{\gamma+1}(r^{\gamma+1} - m_2^{\gamma+1}).$$

Setting $\overline{r} = 2^{\frac{1}{\gamma+1}} m_2$ and taking $r > \overline{r}$,

$$-r^\alpha \phi(|u'(r)|)u'(r) \ge \frac{\lambda f(u(r))}{2(\gamma+1)} r^{\gamma+1},$$

which combined with (5.1) gives

$$-u'(r) \ge \min\left\{\left(\frac{\lambda f(u(r))}{2(\gamma+1)} r^{\gamma-\alpha+1}\right)^{\frac{1}{\gamma_1-1}}, \left(\frac{\lambda f(u(r))}{2(\gamma+1)} r^{\gamma-\alpha+1}\right)^{\frac{1}{\gamma_2-1}}\right\},\ r > \overline{r}. \tag{4.34}$$

Integrating (4.34) from $\overline{r}$ to r and making the change of variables $u(t) = s$, we get

$$\int_{u(\overline{r})}^{u(r)} -\max\{f(t)^{\frac{-1}{\gamma_1-1}}, f(t)^{\frac{-1}{\gamma_2-1}}\}dt \ge \int_{\overline{r}}^r \min\left\{\left(\frac{t^{\gamma-\alpha+1}}{2(\gamma+1)}\right)^{\frac{1}{\gamma_1-1}}, \left(\frac{t^{\gamma-\alpha+1}}{2(\gamma+1)}\right)^{\frac{1}{\gamma_2-1}}\right\} dt.$$

Taking lim inf in both sides, we arrive at a contradiction with (f_3) and so (4.33) is true. To finish the proof of (4.4) we argue as in (4.30) and (4.33) to get zeroes z_4, z_5 and inductively, a sequence with the properties asserted in (1.4).

Proof of (1.5). We start by proving that $z_1(d) \to 0$ when $d \to 0$. By (4.22) and (4.24) we obtain

$$-u'(r) = h^{-1}\left(r^{-\alpha}\int_0^r \lambda t^\gamma f(u(t))dt\right),\ r \in [0, z_1].$$

Now we apply (f_2) and Lemma 5.1 to conclude that

$$-u'(r) \ge \min\left\{\left(\lambda\frac{r^{\gamma-\alpha+1} f(u(r))}{\gamma+1}\right)^{\frac{1}{\gamma_1-1}}, \left(\lambda\frac{r^{\gamma-\alpha+1} f(u(r))}{\gamma+1}\right)^{\frac{1}{\gamma_2-1}}\right\},\ r \in [0, z_1],$$

which implies that

$$-u'(r)\max\left\{f(u(r))^{\frac{-1}{\gamma_1-1}}, f(u(r))^{\frac{-1}{\gamma_2-1}}\right\} \ge \min\left\{\left(\lambda\frac{r^{\gamma-\alpha+1}}{\gamma+1}\right)^{\frac{1}{\gamma_1-1}}, \left(\lambda\frac{r^{\gamma-\alpha+1}}{\gamma+1}\right)^{\frac{1}{\gamma_2-1}}\right\},\ r \in [0, z_1].$$

Integrating from 0 to r and making the change of variables $y = u(t)$ we get to

$$\int_{u(r)}^d \max\left\{f(t)^{\frac{-1}{\gamma_1-1}}, f(t)^{\frac{-1}{\gamma_2-1}}\right\} dt \ge \int_0^r \min\left\{\left(\lambda\frac{t^{\gamma-\alpha+1}}{\gamma+1}\right)^{\frac{1}{\gamma_1-1}}, \left(\lambda\frac{t^{\gamma-\alpha+1}}{\gamma+1}\right)^{\frac{1}{\gamma_2-1}}\right\} dt.$$

Taking $r = z_1(d)$ in the previous inequality and making use of (γ, α) and (f_3), we conclude that $z_1(d) \to 0$ as $d \to 0$. Now, letting $\ell \geq 1$, we assume that $u(r) > 0$ in $(z_\ell(d), z_{\ell+1}(d))$, so that by the notations of (4.30) and (4.33) we have $u'(r) > 0$ in $(z_\ell(d), m_\ell(d))$ and $u'(r) < 0$ in $(m_l(d), z_{\ell+1}(d))$ (the case $u(r) < 0$ in $(z_\ell(d), z_{\ell+1}(d))$ is handled similarly). Now, using (f_2) in (4.22), taking $m_\ell(d) \leq r \leq z_{\ell+1}(d)$ and then applying lemma 5.1, we obtain successively

$$r^\alpha h(-u'(r)) \geq \lambda f(u(r)) \frac{r^{\gamma+1} - m_\ell(d)^{\gamma+1}}{\gamma+1},$$

$$-u'(r) \max\{f(u(r))^{\frac{-1}{\gamma_1-1}}, f(u(r))^{\frac{-1}{\gamma_2-1}}\} \geq \min\left\{\left(\lambda \frac{r^{\gamma+1} - m_\ell(d)^{\gamma+1}}{(\gamma+1)r^\alpha}\right)^{\frac{1}{\gamma_1-1}}, \left(\lambda \frac{r^{\gamma+1} - m_\ell(d)^{\gamma+1}}{(\gamma+1)r^\alpha}\right)^{\frac{1}{\gamma_2-1}}\right\}.$$

Note that $r^{\gamma-\alpha} \geq m_\ell(d)^{\gamma-\alpha}$ since $\gamma \geq \alpha$, therefore

$$r^{\gamma-\alpha+1} - r^{-\alpha} m_\ell(d)^{\gamma+1} \geq m_\ell(d)^{\gamma-\alpha}(r - m_\ell(d)),$$

which gives

$$\begin{aligned} &-u'(r) \max\{f(u(r))^{\frac{-1}{\gamma_1-1}}, f(u(r))^{\frac{-1}{\gamma_2-1}}\} \geq \\ &\min\left\{\left[\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma+1)}(r - m_\ell(d))\right]^{\frac{1}{\gamma_1-1}}, \left[\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma+1)}(r - m_\ell(d))\right]^{\frac{1}{\gamma_2-1}}\right\}. \end{aligned} \tag{4.35}$$

Integrating from $m_\ell(d)$ to $z_{\ell+1}(d)$, making the change of variables $y = u(t)$, we find that

$$\begin{aligned} &\int_0^{u(m_\ell(d))} \max\{f(t)^{\frac{-1}{\gamma_1-1}}, f(t)^{\frac{-1}{\gamma_2-1}}\} dt \geq \\ &\int_{m_\ell(d)}^{z_{\ell+1}(d)} \min\left\{\left[\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma+1)}(r - m_\ell(d))\right]^{\frac{1}{\gamma_1-1}}, \left[\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma+1)}(r - m_\ell(d))\right]^{\frac{1}{\gamma_2-1}}\right\} dt. \end{aligned} \tag{4.36}$$

Assume now $z_\ell(d) < r < m_\ell(d)$. Then by a similar argument, this time, integrating from $z_\ell(d)$ to $m_\ell(d)$ we deduce that

$$\begin{aligned} &\int_0^{u(m_\ell(d))} \max\{f(t)^{\frac{-1}{\gamma_1-1}}, f(t)^{\frac{-1}{\gamma_2-1}}\} dt \geq \\ &\int_{z_\ell(d)}^{m_\ell(d)} \min\left\{\left[\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma+1)}(m_\ell(d) - r)\right]^{\frac{1}{\gamma_1-1}}, \left[\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma+1)}(m_\ell(d) - r)\right]^{\frac{1}{\gamma_2-1}}\right\} dt. \end{aligned} \tag{4.37}$$

Now, since $u(m_\ell(d)) \leq d$ we have by (f_3) that the left hand side of (4.36) and (4.37) converge to zero, and therefore, $\lim_{d\to 0} z_\ell(d) = \lim_{d\to 0} z_{\ell+1}(d)$ for each $\ell \geq 1$. Once $z_1(d) \to 0$ as $d \to 0$, we see that $z_\ell(d) \to 0$ as $d \to 0$.

We pass to the proof that $z_\ell(d) \to z_\ell(d_0)$ if $d \to d_0$. Let us first show that $z_1(d) \to z_1(d_0)$ as $d \to d_0$. Indeed, let $d_n \to d_0$, $u_n(\cdot) = u(\cdot, d_n)$ and $u_0(\cdot) = u(\cdot, d_0)$ so that we have from (4.3) that $u_n \to u$ uniformly in compact subsets of $(0, \infty)$. For each $\epsilon > 0$ small we find

$$u_0(r) > 0, \ 0 \leq r \leq z_1(d_0) - \epsilon \text{ and } u_0(z_1(d_0) + \epsilon) < 0,$$

so that

$$u_n(r) > 0,\ 0 \le r \le z_1(d_0) - \epsilon \text{ and } u_n(z_1(d_0) + \epsilon) < 0,$$

for sufficiently large n. As a consequence, $z_1(d_0) - \epsilon < z_1(d_n) < z_1(d_0) + \epsilon$, showing that $z_1(d_n) \to z_1(d_0)$. Now, assume by induction that $z_\ell(d_n) \to z_\ell(d_0)$ for some $\ell > 1$. We will show that $z_{\ell+1}(d_n) \to z_{\ell+1}(d_0)$. For that matter, we assume $u_0(t) < 0$ for $z_\ell(d_0) < t < z_{\ell+1}(d_0)$ (the other case is handled similarly). Taking $\epsilon > 0$ small, we find that $u_n(t) < 0$ for $z_\ell(d_0) + \epsilon \le t \le z_{\ell+1}(d_0) - \epsilon$ and $u_n(z_{\ell+1}(d_0) + \epsilon) > 0$, showing that $z_{\ell+1}(d_0) - \epsilon < z_{\ell+1}(d_n) < z_{\ell+1}(d_0) + \epsilon$. Consequently, $z_{\ell+1}(d_n) \to z_{\ell+1}(d_0)$ as $d \to d_0$, which finishes the proof of (1.5).

Proof of (1.6).

Let $d \in (0, d_0)$. It suffices to show that $z_{\ell+2}(d) > R$ whenever d is close enough to d_0. We assume that $u(r, d_0) < 0$ for $r \in (z_\ell(d_0), z_{\ell+1}(d_0))$ (the other case is handled similarly).

Notice that as $z_\ell(d_0)$ is increasing an there is only ℓ zeroes in $(0, R)$, we must show that $z_{\ell+1}(d_0) \ge R$ and $z_{\ell+2}(d_0) > R$. However, as $z_{\ell+2}(d) \to z_{\ell+2}(d_0)$ for $d \to d_0$, we have $z_{\ell+2}(d) > R$ whenever d is close enough to d_0. This completes the proof of Theorem 1.2. $\square$

5 Appendix

Remark 5.1. *(***On the radially symetric form of** (Φ)*) Let u be a weak solution of (Φ), radially symmetric in the sense that $u(x) = u(|x|) = u(r)$. Let $r \in (0, R)$ and pick $\epsilon > 0$ small such that $0 < r < r + \epsilon < R$.*

Consider the radially symmetric cut-off function $v_{r,\epsilon}(x) = v_{r,\epsilon}(r)$, where

$$v_{r,\epsilon}(t) := \begin{cases} 1 \quad if \ \ 0 \le t \le r, \\ linear \quad if \ \ r \le t \le r + \epsilon, \\ 0 \quad if \ \ r + \epsilon \le t \le R. \end{cases}$$

and notice that $v_{r,\epsilon} \in W_0^{1,\Phi}(B)| \cap Lip(\overline{B})$. By replacing v with $v_{r,\epsilon}$ in (2.2), we get to

$$\frac{-1}{\epsilon} \int_{B(0,r+\epsilon)\setminus B(0,r)} \phi(|u'(|x|)|)u'(|x|)dx = \lambda \int_{B(0,r+\epsilon)} f(u(|x|))v_{r,\epsilon}(|x|)dx.$$

Making the change of variables $x = r\omega$ with $r > 0$ and $\omega \in \partial B(0, 1)$ and letting $\epsilon \to 0$ we infer that

$$\phi(|u'(r)|)u'(r)r^{N-1} = \lambda \int_0^r f(u(r))r^{N-1}dr,$$

which gives

$$(r^{N-1}\phi(|u'(r)|)u'(r))' = \lambda r^{N-1} f(u(r)).$$

So the radially symetric form of (Φ) is

$$\begin{cases} -(r^{N-1}\phi(|u'(r)|)u'(r))' &= \lambda r^{N-1} f(u(r),\ 0 < r < R \\ \\ \qquad u'(0) = u(R) = 0. \end{cases}$$

Lemma 5.1. *Assume that ϕ satisfies (ϕ_1)-(ϕ_3). Then*

$$h(1)\min\{h^{-1}(s)^{\gamma_1 - 1}, h^{-1}(s)^{\gamma_2 - 1}\} \le s \le h(1)\max\{h^{-1}(s)^{\gamma_1 - 1}, h^{-1}(s)^{\gamma_2 - 1}\},\ s > 0.$$

Proof. Condition (ϕ_3) implies that

$$(\gamma_1 - 1)\frac{d}{dt}\ln t \le \frac{d}{dt}\ln h(t) \le (\gamma_2 - 1)\frac{d}{dt}\ln t,\ \forall\ t > 0.$$

Let $t \le 1$. Integrating the previous inequality from t to 1, we get

$$h(1)t^{\gamma_1 - 1} \le h(t) \le h(1)t^{\gamma_2 - 1},\ t \le 1.$$

Let $t \ge 1$. Iintegrating the previous inequality from 1 to t, we get

$$h(1)t^{\gamma_2 - 1} \le h(t) \le h(1)t^{\gamma_1 - 1},\ \forall\ t \ge 1.$$

Therefore

$$h(1)\min\{t^{\gamma_1 - 1}, t^{\gamma_2 - 1}\} \le h(t) \le h(1)\max\{t^{\gamma_1 - 1}, t^{\gamma_2 - 1}\},\ \forall\ t > 0.$$

Letting $t = h^{-1}(s)$, the lemma is proved. □

Lemma 5.2. *Assume ϕ satisfies (ϕ_1)-(ϕ_3). Then*

$$\Phi(1)\min\{t^{\gamma_1}, t^{\gamma_2}\} \le \Phi(t) \le \Phi(1)\max\{t^{\gamma_1}, t^{\gamma_2}\},\ t > 0.$$

Proof. From (ϕ_3),

$$\gamma_1 t\phi(t) \le th'(t) + t\phi(t) \le \gamma_2 t\phi(t),\ \forall t > 0,$$

which implies, after integration from 0 to t that,

$$\gamma_1 \le \frac{t\Phi'(t)}{\Phi(t)} \le \gamma_2,\ t > 0. \tag{5.1}$$

The previous inequality is called condition Δ_2. To finish the proof, we proceed as in the proof of lemma 5.1 to conclude the desired inequality. □

Lemma 5.3. *Assume that ϕ satisfies (ϕ_1)-(ϕ_3). Then*

$$[h^{-1}]'(t) \le \frac{t^{\frac{-\gamma_2+2}{\gamma_2-1}}}{h(1)^{\gamma_2}(\gamma_1 - 1)},\ t \le 1.$$

Proof. Remember that

$$[h^{-1}]'(t) = \frac{1}{h'(h^{-1}(t))},\ t > 0. \tag{5.2}$$

From the proofs of Lemmas 5.1 and 5.2,

$$h(1)(\gamma_1 - 1)\min\{t^{\gamma_1 - 2}, t^{\gamma_2 - 2}\} \le h'(t) \le h(1)(\gamma_2 - 1)\max\{t^{\gamma_1 - 2}, t^{\gamma_2 - 2}\} \text{ for } t > 0. \tag{5.3}$$

Gathering (5.2) and (5.3), we see that

$$[h^{-1}]'(t) \le \frac{[h^{-1}(t)]^{-\gamma_2+2}}{h(1)(\gamma_1 - 1)},\ t \le 1.$$

Now we use Lemma 5.1 to obtain

$$[h^{-1}]'(t) \le \frac{t^{\frac{-\gamma_2+2}{\gamma_2-1}}}{h(1)^{\gamma_2}(\gamma_1 - 1)},\ t \le 1.$$

□

Lemma 5.4. *Assume that $\phi : (0,\infty) \to (0,\infty)$ is a differentiable function satisfying (ϕ_3). Then, there is a positive constant Γ_1 such that*

$$\sum_{i,j=1}^{N} \frac{\partial a_j}{\partial \eta_i}(\eta)\xi_i\xi_j \geq \Gamma_1 \phi(|\eta|)|\xi|^2, \tag{5.4}$$

where $a_j(\eta) = \phi(|\eta|)\eta_j$, $\eta \in \mathbb{R}^N \setminus \{0\}$ and $\xi \in \mathbb{R}^N$.

Proof. Indeed, by (ϕ_3),

$$(\gamma_1 - 2)\phi(t) \leq t\phi'(t) \leq (\gamma_2 - 2)\phi(t). \tag{5.5}$$

Suppose first that $\gamma_1 < 2$. Note that

$$\sum_{i,j=1}^{N} \frac{\partial a_j}{\partial \eta_i}(\eta)\xi_i\xi_j = \phi(|\eta|)|\xi|^2 + \frac{\phi'(|\eta|)|\langle \eta, \xi\rangle|^2}{|\eta|} \tag{5.6}$$

If $\phi'(|\eta|) < 0$, then $\phi'(|\eta|)|\langle \eta, \xi\rangle|^2 \geq \phi'(|\eta|)|\eta|^2|\xi|^2$. From (5.5) and (5.6),

$$\sum_{i,j=1}^{N} \frac{\partial a_j}{\partial \eta_i}(\eta)\xi_i\xi_j \geq (\gamma_1 - 1)\phi(|\eta|)|\xi|^2.$$

If $\phi'(|\eta|) \geq 0$, then take $\Gamma_1 = 1$.

If $\gamma_1 \geq 2$, then (5.5) is satisfied with $\Gamma_1 = 1$, as can readily be seen from (5.6) and noting that $\phi'(t) \geq 0$ in this case. □

We now prove a Simon type inequality.

Lemma 5.5. *Assume that $\phi : (0,\infty) \to (0,\infty)$ is a differentiable function satisfying (ϕ_1)-(ϕ_3). Then*

$$\langle \phi(|\eta|)\eta - \phi(|\eta'|)\eta', \eta - \eta'\rangle \geq \min\{4, 4\Gamma_1\} \frac{|\eta - \eta'|}{1 + |\eta| + |\eta'|} \Phi\left(\frac{|\eta - \eta'|}{4}\right), \tag{5.7}$$

where Γ_1 was given in lem.a 5.4, $\eta, \eta' \in \mathbb{R}^N$ and $\langle \cdot, \cdot\rangle$ denotes inner product.

Proof. If $\eta, \eta' = 0$ then (5.7) is obviously satisfied. If only one of them is 0, let's say, $\eta' = 0$, then

$$\phi(|\eta|)|\eta|^2 \geq \Phi(|\eta|) \geq 4\Phi\left(\frac{|\eta|}{4}\right),$$

where in the last inequalities we have used the properties of an N-function (note that an N-function is convex). So (5.7) is satisfied. If $\eta, \eta' \neq 0$, assume without loss of generality that $|\eta| \leq |\eta'|$. Then, an application of Cauchy-Schwartz inequality implies that

$$\frac{|\eta - \eta'|}{4} \leq |t\eta + (1-t)\eta'| \leq 1 + |\eta| + |\eta'|, \ t \in [0, 1/4].$$

We conclude from the last inequality, (5.6) and the properties of an N-function that

$$\begin{aligned}
\langle \phi(|\eta|)\eta - \phi(|\eta'|)\eta', \eta - \eta' \rangle &= \sum_{i=1}^{N} \int_0^1 \frac{d}{dt}[a_j(t\eta + (1-t)\eta')](\eta_j - \eta_j')dt \\
&= \int_0^1 \sum_{i,j=1}^{N} \frac{\partial a_j}{\partial \eta_i}[t\eta + (1-t)\eta'](\eta_i - \eta_i')(\eta_j - \eta_j')dt \\
&\geq \Gamma_1 \int_0^1 \phi(|t\eta + (1-t)\eta'|)|\eta - \eta'|^2 dt \\
&\geq \Gamma_1 \int_0^{1/4} \phi(|t\eta + (1-t)\eta'|)|\eta - \eta'|^2 dt \\
&= \Gamma_1 \int_0^{1/4} \phi(|t\eta + (1-t)\eta'|)|\eta - \eta'|^2 \frac{|t\eta + (1-t)\eta'|}{|t\eta + (1-t)\eta'|} dt \\
&\geq 4\Gamma_1 \frac{|\eta - \eta'|}{1 + |\eta| + |\eta'|} \phi\left(\frac{|\eta - \eta'|}{4}\right)\left(\frac{|\eta - \eta'|}{4}\right)^2 \\
&\geq 4\Gamma_1 \frac{|\eta - \eta'|}{1 + |\eta| + |\eta'|} \Phi\left(\frac{|\eta - \eta'|}{4}\right).
\end{aligned}$$

□

Lemma 5.6. *Assume ϕ satisfies (ϕ_1)-(ϕ_3). Then, the function $H(t) = t\Phi'(t) - \Phi(t)$ is strictly increasing and satisfies*

$$(\gamma_1 - 1)\Phi(t) \leq H(t) \leq (\gamma_2 - 1)\Phi(t),\ t \geq 0,$$

$$\frac{\gamma_1 - 1}{\gamma_1} t\Phi'(t) \leq H(t) \leq \frac{\gamma_2 - 1}{\gamma_2} t\Phi'(t),\ t \geq 0.$$

Proof. Indeed, as (ϕ_3) is satisfied, we have that

$$t\Phi'(t) - r\Phi'(r) > (t-r)\Phi'(t) > \int_r^t \tau\phi(\tau)d\tau, t > r \geq 0,$$

which implies that H is strictly incresing. On the other hand, condition (5.1) implies the desired inequalities.

□

Proof of Lemma 4.3 Indeed, take $\rho \in (0, d)$ and set

$$K_\rho^\epsilon(d) = \{u \in C([0,\epsilon]) \mid u(0) = d,\ \|u - d\|_\infty \leq \rho\}.$$

Take $\epsilon > 0$ small. If $u \in K_\rho^\epsilon(d)$, then by continuity, $u(r) > 0$, $r \in [0,\epsilon]$. Hence, for small ϵ, a solution of $(P_{\lambda,d,\epsilon})$ satisfies $u'(r) \leq 0$ for $r \in [0,\epsilon]$ (this was showed in the proof of proposition (4.1)) and

$$u(r) = d - \int_0^r h^{-1}\left(t^{-\alpha}\int_0^t \lambda\tau^\gamma f(u(\tau))d\tau\right)dt,\ \forall\ r \in [0,\epsilon].$$

We infer that the solutions of $(P_{\lambda,d,\epsilon})$, for small ϵ, are fixed points of the operator

$$T(u(r)) = d - \int_0^r h^{-1}\left(t^{-\alpha}\int_0^t \lambda\tau^\gamma f(u(\tau))d\tau\right)dt,\ \forall\ r \in [0,\epsilon].$$

Now we will verify that there exist $\epsilon, \rho > 0$ and $k \in (0,1)$ such that

$$T\left(K_\rho^\epsilon(d)\right) \subset K_\rho^\epsilon(d), \tag{5.8}$$

and

$$\|Tu - Tv\|_\infty \leq k\|u - v\|_\infty. \tag{5.9}$$

Therefore, by the Banach Fixed Point Theorem, T has a unique fixed point, which in turn will be a $C^2([0,\epsilon])$ solution of $(P_{\lambda,d,\epsilon})$. With respect to (5.8), let $\rho \in (0, d/2]$, which implies that $u(r) \in [d/2, 2d]$ for $u \in K_\rho^\epsilon(d)$. Therefore, for $u \in K_\rho^\epsilon(d)$ we have that

$$h^{-1}\left(r^{-\alpha}\int_0^s \lambda t^\gamma f(u(t))dt\right) \leq h^{-1}\left(\frac{\lambda\|f\|_{\infty,d}s^{\gamma-\alpha+1}}{\gamma+1}\right), \ s \in [0,\epsilon],$$

where $\|f\|_{\infty,d} = \max\limits_{s\in[d/2,2d]} f(s)$. For small ϵ, we can apply lemma 5.1 in the Appendix to conclude from the previous inequality that

$$\begin{aligned}
|T(u(r)) - T(u(0))| &= \int_0^r h^{-1}\left(r^{-\alpha}\int_0^s \lambda t^\gamma f(u(t))dt\right) ds \\
&\leq \int_0^r h^{-1}\left(\frac{\lambda\|f\|_{\infty,d}s^{\gamma-\alpha+1}}{\gamma+1}\right) ds \\
&\leq \int_0^r h(1)\left(\frac{\lambda\|f\|_{\infty,d}s^{\gamma-\alpha+1}}{\gamma+1}\right)^{\frac{1}{\gamma_1-1}} ds \\
&= h(1)\left(\frac{\lambda\|f\|_{\infty,d}}{\gamma+1}\right)^{\frac{1}{\gamma_1-1}} r^{\frac{\gamma-\alpha+\gamma_1}{\gamma_1-1}}, \ r \in [0,\epsilon].
\end{aligned}$$

As $\gamma \geq \alpha$, we obtain from the last inequality that there is $\epsilon > 0$ such that $Tu \in C([0,\epsilon])$ and $|T(u(r)) - d| \leq \rho$ for $r \in [0,\epsilon]$, which finishes the proof of (5.8). Now we pass to the proof of (5.9). We first prove it by assuming that $f \in C(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{0\})$. Fix ρ as in (5.8) and take $u, v \in K_\rho^\epsilon(d)$. By the Mean Value Theorem, there is $h \in (0,1)$ such that

$$T(v(r)) - T(u(r)) = \int_0^r \left[h^{-1}\left(s^{-\alpha}\int_0^s \lambda t^\gamma f(u(t))dt\right) - h^{-1}\left(s^{-\alpha}\int_0^s \lambda t^\gamma f(v(t))dt\right)\right] ds =$$

$$\int_0^r \left[(h^{-1})'\left(s^{-\alpha}\int_0^s \lambda t^\gamma f(hu(t) + (1-h)v(t))dt\right)\left(s^{-\alpha}\int_0^s \lambda t^\gamma f'(hu(t) + (1-h)v(t))(u(t) - v(t))dt\right)\right] ds.$$

Choose ϵ small in such a way that the number $s^{-\alpha}\int_0^s \lambda t^\gamma f(hu(t) + (1-h)v(t))dt$ for $s \in [0,\epsilon]$ is small. Therefore, Lemma 5.3 and the last equality implies that for $1 < \gamma_2 \leq 2$ (note that in this case, the function $t \mapsto t^{\frac{-\gamma_2+2}{\gamma_2-1}}$ is decreasing)

$$|T(v(r))-T(u(r))|\leq$$

$$\int_0^r\left[c\left(s^{-\alpha}\int_0^s\lambda t^\gamma|f(hu(t)+(1-h)v(t))|dt\right)^{\frac{-\gamma_2+2}{\gamma_2-1}}\left(s^{-\alpha}\int_0^s\lambda t^\gamma|f'(hu(t)+(1-h)v(t))||u(t)-v(t)|dt\right)\right]ds\leq$$

$$\int_0^r\left[c\left(s^{-\alpha}\int_0^s\lambda\|f\|_{\infty,d'}t^\gamma dt\right)^{\frac{-\gamma_2+2}{\gamma_2-1}}\left(s^{-\alpha}\int_0^s\lambda\|f'\|_{\infty,d}t^\gamma dt\right)\|u-v\|_\infty\right]ds=$$

$$\int_0^r\left[c\left(\lambda\|f\|_{\infty,d'}\frac{s^{-\alpha+\gamma+1}}{\gamma+1}\right)^{\frac{-\gamma_2+2}{\gamma_2-1}}\left(\lambda\|f'\|_{\infty,d}\frac{s^{-\alpha+\gamma+1}}{\gamma+1}dt\right)\|u-v\|_\infty\right]ds=$$

$$c\left(\frac{\lambda\|f\|_{\infty,d'}}{\gamma+1}\right)^{\frac{-\gamma_2+2}{\gamma_2-1}}\frac{\lambda\|f'\|_{\infty,d}}{\gamma+1}r^{\frac{-\alpha+\gamma+\gamma_2}{\gamma_2-1}}\|u-v\|_\infty,$$

where $\|f\|_{\infty,d'}=\min_{s\in[d/2,2d]}|f(s)|$ and $\|f'\|_{\infty,d}=\max_{s\in[d/2,2d]}|f'(s)|$. If on the other hand, we have that $\gamma_2\geq 2$, i.e., $t\mapsto t^{\frac{-\gamma_2+2}{\gamma_2-1}}$ is increasing then, we must conclude that

$$|T(v(r))-T(u(r))|\leq c\left(\frac{\lambda\|f\|_{\infty,d}}{\gamma+1}\right)^{\frac{-\gamma_2+2}{\gamma_2-1}}\frac{\lambda\|f'\|_{\infty,d}}{\gamma+1}r^{\frac{-\alpha+\gamma+\gamma_2}{\gamma_2-1}}\|u-v\|_\infty,$$

where $\|f\|_{\infty,d}=\max_{\in[d/2,d]}f(s)$. In both cases, hypothesis (γ,α) implies the existence of ϵ such that (5.9) is true in the case $f\in C(\mathbb{R})\cap C^1(\mathbb{R}\setminus\{0\})$.

$\square$

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